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(FOR THE CANDIDATES ADMITTED DURING THE ACADEMIC YEAR 2023 ONLY)

23PMS416

REG.NO.:

N.G.M. COLLEGE (AUTONOMOUS): POLLACHI

END-OF-SEMESTER EXAMINATIONS: MAY-2025

M.Sc.-MATHEMATICS **MAXIMUM MARKS: 75**

SEMESTER: IV **TIME: 3 HOURS**

OPERATOR THEORY

SECTION – A (10 X 1 = 10 MARKS)

ANSWER THE FOLLOWING QUESTIONS.
MULTIPLE CHOICE QUESTIONS. **(K1)**

1. The norm of the linear space l^p(n) is $\|x\| = (\sum_{i=1}^{n}|x(i)|^p)^{1/p}$ if _____.

   a. $1 \leq p < \infty$ b. $1 < p < \infty$ c. $1 \leq p \leq \infty$ d. $1 \leq p \leq \infty$

2. Every bounded sequence in a reflexive space has _____ subsequence.

   a. no weakly convergent b. weakly closed

   c. a weakly convergent d. a weak-star convergence

3. $\|xy\|$_____$\|x\|\ \|y\|$

   a. $\leq$ b. $>$ c. $=$ d. $\geq$

4. _____ is also known as point spectrum.

   a. Eigenvector b. Eigenspace

   c. Eigenspectrum d. Eigenvalue

5. Let X be a Hilbert space and $A \in B(X)$. Then A is said to be a unitary operator if _____.

   a. $A^*A = I = AA^*$ b. $A^*A = AA^*$ c. $A^* = A$ d. $A^* = I = A$

ANSWER THE FOLLOWING IN ONE (OR) TWO SENTENCES **(K2)**

6. State Fundamental theorem of Lebesgue integration.
7. Define Reflexive.
8. Define completely continuous operator.
9. Define spectral radius.
10. Define adjoint of a linear operator.

SECTION – B (5 X 5 = 25 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS. **(K3)**

11. a) Let $1 < p < \infty$ and for each $j \in \mathbb{N}$, let $f_j(x) = x(j)$ for $x \in l^p$. Then, prove that $f_j \in (l^p)'$ for all $j \in \mathbb{N}$, and $\{f_j : j \in \mathbb{N}\}$ is a Schauder basis of $(l^p)'$.
    In fact, $f = \sum_{j=1}^{\infty} f(e_j)\, f_j,\ \forall f \in (l^p)'$.

    (OR)

    b) Let $u_a = 0$, and for $a < t \leq p$, let $u_t = \chi_{(a,t]}$. Then, prove that for every f in the dual of $L^p[a,b],\ 1 \leq p < \infty$, the function v: $[a,b] \to \mathbb{K}$ defined by $v(t) = f(u_t),\ t \in [a,b]$, is absolutely continuous.

(CONTD……..2)

12. a) Let $1 < p < \infty$, and let (x_n) be a bounded sequence in l^p such that, for every $j \in \mathbb{N}, x_n(j) \to x(j)$ for some $x \in l^p$. Then prove that (x_n) converges weakly to x.

(OR)

b) Suppose X is a reflexive space. Then prove that if Y is a closed subspace of X, then for every $x \in X$, there exists $y \in Y$ such that $\|x - y\| = \text{dist}(x, Y)$.

13. a) Let X be a normed linear space, Y be a Banach space and $A \in B(X, Y)$. If (A_n) is a sequence in $\mathcal{K}(X, Y)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$, then prove that $A \in \mathcal{K}(X, Y)$.

(OR)

b) Let X and Y be normed linear spaces and $A: X \to Y$ be a linear operator. If $A \in \mathcal{K}(X, Y)$, then prove that for every sequence (x_n) in X, $x_n \to x$ weakly $Ax_n \to Ax$.

14. a) Let X be a normed linear space and $A \in B(X)$. Then prove that $\sigma_{app}(A)$ is a closed subset of $\mathbb{K}$.

(OR)

b) Let X be a Banach space, $p(t)$ be a polynomial with coefficients in $\mathbb{K}$ and $A \in B(X)$. Then prove that $\{p(\lambda): \lambda \in \sigma(A)\} \subseteq \sigma(p(A))$.
If $\mathbb{K} = \mathbb{C}$, then prove that $\{p(\lambda): \lambda \in \sigma(A)\} = \sigma(p(A))$.

15. a) State and prove the existence of the adjoint for linear operators.

(OR)

b) Let X be a Hilbert space and $A \in B(X)$. Then prove that
(i) A is normal if and only if $\|Ax\| = \|A^*x\|$ for every $x \in X$,
(ii) A is unitary if and only if A is surjective and $\|Ax\| = \|x\|$ for every $x \in X$
In particular, if A is a unitary operator, then prove that $\|A\| = 1$.

SECTION – C (5 X 8 = 40 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.

(K4 (Or) K5)

16. a) Prove that the map $T: (l^p(n))' \to l^q(n)$ defined by $T(f) = (f(e_1), \ldots . f(e_n)), f \in (l^p)'$, is a surjective linear isometry, where $e_j \in \mathbb{K}^n$ is such that $e_j(i) = \delta_{ij}$ for $i, j = 1,2, \ldots, n$.

(OR)

b) Let $C[a, b]$ be with $\|\cdot\|_\infty$. For $\phi \in NBV[a, b]$, let $f_\phi(x) = \int_a^b x \, d\phi \, , x \in C[a, b]$. Then prove that $f_\phi \in (C[a, b])'$, and the map $\phi \mapsto f_\phi$ is a linear isometry from $NBV[a, b]$ onto the dual of $C[a, b]$.

17. a) Prove that (i) Every closed subspace of a reflexive space is reflexive.
(ii) Every normed linear space which is linearly isometric with a reflexive space is reflexive.

(OR)

b) State and prove Schur's lemma.

18. a) Let X and Y be normed linear spaces and $A: X \to Y$ be a linear operator. Then prove that the following are equivalent:
(i) A is a compact operator.
(ii) $cl\{Ax: \|x\| < 1\}$ is compact in Y.
(iii) For every bounded subset E of X, $cl\, A(E)$ is compact in Y.
(iv) For every bounded sequence (x_n) in X, the sequence (Ax_n) has a convergent subsequence in Y.

(CONTD……..3)

(OR)

18. b) Let X and Y be normed linear spaces. Then prove that the following:
(i) If $A \in \mathcal{K}(X,Y)$, then $A' \in \mathcal{K}(Y',X')$.
(ii) Converse of (i) holds if Y is a Banach space.

19. a) Let X be a normed linear space and $A: X \to Y$ be a compact operator. Then prove that $\sigma_{eig}(A)$ is a countable subset of $\mathbb{K}$, and zero is the only possible limit point of $\sigma_{eig}(A)$.

(OR)

b) State and prove Spectral radius formula.

20. a) State and prove closed range theorem.

(OR)

b) Let X be a Hilbert space and $A \in B(X)$ be a self-adjoint operator. Then prove that $\|A\| = sup\{|\langle Ax, x\rangle| : x \in X,\ \|x\| = 1\}$. In particular, prove that $A = 0$ if and only if $\langle Ax, x\rangle = 0$ for all $x \in X$.

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