(NO. OF PAGES: 3)

(FOR THE CANDIDATES ADMITTED DURING THE ACADEMIC YEAR 2024 ONLY)

24PMS205

REG.NO. :

N.G.M.COLLEGE (AUTONOMOUS) : POLLACHI

END-OF-SEMESTER EXAMINATIONS : MAY-2025

M.Sc.-MATHEMATICS **MAXIMUM MARKS: 75**

SEMESTER: I **TIME : 3 HOURS**

PART - III

LINEAR ALGEBRA

SECTION – A (10 X 1 = 10 MARKS)

ANSWER THE FOLLOWING QUESTIONS.

MULTIPLE CHOICE QUESTIONS. **(K1)**

1. The characteristic polynomial of A = $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ is_______________.

   (a) $(x-1)(x-2)^2$ (b) $(x-1)^2(x-2)$ (c) $(x-3)(x-1)^2$ (d) $x(x-3)(x-2)$

2. N is a nilpotent operator if ______________.

   (a) There exist a positive integer r such that $N^{r-1} = 0$.

   (b) There exist a positive integer r such that $rN = 0$.

   (c) There exist a positive integer r such that $N^r = 0$.

   (d) For every positive integer r such that $N^r = 0$.

3. The space $Z(\alpha;T)$ is one-dimensional if and only if _______.

   (a) $\alpha = 0$ (b) α is a characteristic vector for T *(c)* α is not a characteristic vector for T *(d)* α =

4. How many possible Jordan forms are there for a 6 X 6 complex matrix with characteristic polynomial $(x+2)^4(x-1)^2$. ______________.

   (a) 2 (b) 5 (c) 7 (d) 10

5. Let f be a bilinear form on the vector space V. f is symmetric if _____ for all α, β in V.

   (a) $f(\alpha+\beta) = f(\beta+\alpha)$ (b) $f(\alpha, \beta) = f(\beta,\alpha)$ (c) $f(\alpha\beta) = f(\beta\alpha)$ (d) $f(\alpha, \beta) = - f(\beta,\alpha)$

ANSWER THE FOLLOWING IN ONE (OR) TWO SENTENCES. **(K2)**

6. Define minimal polynomial for the linear operator T on a finite dimensional vector space V (F).
7. Let $W_1, \ldots, W_k$ be subspaces of the vector space V. When we say $W_1, W_2, \ldots, W_k$ are independent?
8. If T is a nilpotent linear on a vector space of dimension n, then write the characteristic equation for T.
9. When we say a matrix is in normal form?
10. Define: Symmetric bilinear form.

(CONTD……..2)

SECTION – B (5 X 5 = 25 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS. (K3)

11. a) Let $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Find a matrix P such $P^{-1}AP = D$, where D is the diagonal matrix.

(OR)

b) Let W be an invariant subspace for T. Prove that the characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T and the minimal polynomial for T_w divides the minimal polynomial for T.

12.a) Let V be a finite dimensional vector space. Let $W_1, \ldots, W_k$ be subspaces of V and let $W = W_1 + \ldots + W_k$. Show that the following are equivalent.

(i) $W_1, \ldots, W_k$ are independent.

(ii) **Error! Digit expected.**

(OR)

b) Prove that if *E* is the projection on *R* along *N*, then *(I-E)* is the projection on *N* along *R*.

13.a) Let U be a linear operator on the finite-dimensional space W. Then prove that U has a cyclic vector if and only if there is some ordered basis for W in which U is represented by the companion matrix of the minimal polynomial for U.

(OR)

b) Prove that if T is a linear operator on a finite-dimensional vector space, then every T-admissible subspace has a complementary subspace which is also invariant under T.

14.a) Let A be the complex 3X3 matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}$. Find a, b, c if A is similar to a diagonal matrix.

(OR)

b) Let A be an n x n matrix with entries in the field F, and let $p_1, \ldots, p_r$ be the invariant factors for A. Show that the matrix *xI-A* is equivalent to the n x n diagonal matrix with diagonal entries $p_1, \ldots, p_r, 1, 1, \ldots, 1$.

15.a) Let V be a finite-dimensional vector space over the field F. Prove that for each ordered basis $\mathfrak{B}$ of V, the function which associates with each bilinear form on V its matrix in the ordered basis $\mathfrak{B}$ is an isomorphism of the space L(V, V, F) onto the space of n x n matrices over the field F.

(OR)

b) Let V be a finite-dimensional vector space over a field of characteristic zero, and let f be a symmetric bilinear form on V. Then prove that there is an ordered basis for V in which f is represented by a diagonal matrix.

SECTION – C (5 X 8 = 40 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS.

(K4 (Or) K5)

16. a) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Show that the following statements are equivalent.

(i) c is a characteristic value of T.

(ii) The operator (T-cI) is singular.

(iii) det (T-cI) = 0.

(OR)

b) State and prove Cayley-Hamilton theorem.

(CONTD……..3)

17.a) Let T be a linear operator on a finite-dimensional space V. If T is diagonalizable and if $c_1, \ldots, c_k$ are the distinct characteristic values of T, then show that there exist linear operators $E_1, \ldots, E_k$ on V such that (i) $T = c_1E_1 + \ldots + c_kE_k$;

(ii) $I = E_1 + \ldots + E_k$;

(iii) $E_iE_j = 0$, $i \neq j$;

(iv) $E_i^2 = E_i$

(v) The range of E_i is the characteristic space for T associated with c_i.

(OR)

b) State and prove primary decomposition theorem.

18. a) Let α be any non-zero vector in V and let p_α be the T-annihilator of α. Then prove thatr the following:

(i) The degree of p_α is equal to the dimension of the cyclic subspace $Z(\alpha; T)$.

(ii) If the degree of p_α is k, then the vectors $\alpha, T\alpha, T^2\alpha, \ldots, T^{k-1}\alpha$ form a basis for $Z(\alpha; T)$

(iii) If U is the linear operator on $Z(\alpha; T)$ induced by T, then the minimal polynomial for U is p_α.

(OR)

b) If T is a linear transformation on $M_2(\mathbb{R})$ such that $T(A) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} A$.

Find the matrix representation of T with respect to the standard basis of $M_2(\mathbb{R})$, minimal polynomial and the rational form of the matrix $A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

19.a) If A is a complex 5 x 5 matrix with characteristic polynomial $f = (x-2)^3(x+7)^2$ and minimal polynomial $p = (x-2)^2(x+7)$, what is the Jordan form for A?

(OR)

b) Let P be an m x m matrix with entries in the polynomial algebra $F[x]$. Prove that the following statements are equivalent:

(i) P is invertible.

(ii) The determinant of P is a non-zero scalar polynomial.

(iii) P is row-equivalent to the m x m identity matrix.

(iv) P is a product of elementary matrices.

20.a) Let f be a bilinear form on the finite-dimensional vector space V. Let L_f and R_f be linear transformations from V into V* defined by $(L_f\alpha)(\beta) = f(\alpha,\beta) = (R_f\beta)(\alpha)$. Then show that rank (L_f) = rank(R_f).

(OR)

b) Let V be an n-dimensional vector space over the field of real numbers, and let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ for V in which the matrix of f is diagonal and such that $f(\beta_j, \beta_j) = \pm 1$, $j = 1, \ldots, r$. Furthermore, prove that the number of basis vectors β_j for which $f(\beta_j, \beta_j) = 1$ is independent of the choice of basis.

****