(FOR THE CANDIDATES ADMITTED DURING THE ACADEMIC YEAR 2022 ONLY)

22UMS614

REG.NO.:

N.G.M. COLLEGE (AUTONOMOUS): POLLACHI
END-OF-SEMESTER EXAMINATIONS: MAY-2025

B.Sc.- MATHEMATICS **MAXIMUM MARKS: 50**
SEMESTER: VI **TIME: 3 HOURS**

PART-III
LINEAR ALGEBRA

SECTION – A **(10 X 1 = 10 MARKS)**

ANSWER THE FOLLOWING QUESTIONS.
MULTIPLE CHOICE QUESTIONS. **(K1)**

1) If A is 2x2 matrix such that $A^2 = 0$ then ________________.
   (A) $A = 0$ (B) $A \neq 0$
   (C) Rank $A = 0$ (D) Rank $A \neq 0$

2) If V is a finite-dimensional vector space, then any two bases of V have the ________ number of elements.
   (A) finite (B) infinite (C) same (D) fixed

3) Let the vector spaces V and W have dimensions 8 and 4 respectively over F. Then the dimension of L(V, W) is ___________.
   (A) 32 (B) 2 (C) 4 (D) 12

4) Let A and B be $n \times n$ matrices over the field F. Then B is similar to A over F if there is an invertible $n \times n$ matrix P over F such that B = _______.
   (A) PAP^{-1} (B) $P^{-1}AP$ (C) $P^{-1}A$ (D) AP

5) If A be any $m \times n$ matrix over the field F and T is the linear transformation from F^n into F^n, then rank (T) = ____________________.
   (A) row rank (A) (B) column rank (A) (C) rank (A) (D) all of the above

ANSWER THE FOLLOWING IN ONE OR TWO SENTENCES.

6) Define a matrix is row-reduced.
7) Define a subspace.
8) Define linear transformation.
9) Define linear functional.
10) What is a hyperspace?

SECTION – B **(5 X 3 = 15 MARKS)**

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS. **(K3)**

11) (a) Find the solution of the system of equations $-x_1 + ix_2 = 0, -ix_1 + 3x_2 = 0, x_1 + 2x_2 = 0$.

(OR)

(b) Let e be an elementary row operation and let E be the $m \times m$ elementary matrix $E = e(I)$. Then, prove that for every $m \times n$ matrix A, $e(A) = EA$.

12) (a) Prove that if A is a $m \times n$ matrix over F and B, C are $n \times p$ matrices over F, then $A(dB + C) = d(AB) + AC$ for each scalar d in F.

(OR)

(b) Let R be a non-zero row-reduced echelon matrix. Then prove that the non-zero row vectors of R form a basis for the row space of R.

13) (a) Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \ldots . \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \ldots \ldots, \beta_n$ be any vector in W. Then prove that there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j, j = 1,2, \ldots \ldots, n$.

(OR)

(b) Let V and W be vector spaces over the field F and let T be a linear transformation from V $into$ W. If T is invertible, then the inverse function T^{-1} is a linear transformation from W onto V

14) (a) Let V be a finite-dimensional vector space over the field F, and let $\mathfrak{B} = \{\alpha_1, \ldots \ldots, \alpha_n\}$ and $\mathfrak{B}' = \{\alpha_1', \ldots \ldots, \alpha_n'\}$ be ordered bases for V. Suppose T is a linear operator on V. If $P = \{P_1, \ldots \ldots, P_n\}$ is the $n \times n$ matrix with columns $P_j = [\alpha_j']_{\mathfrak{B}}$, then prove that $[T]_{\mathfrak{B}'} = P^{-1}[T]_{\mathfrak{B}}P$.

(OR)

(b) Find the vectors in the subspace annihilate to the following functionals on R^4:

$f_1(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 2x_3 + x_4,$

$f_2(x_1, x_2, x_3, x_4) = 2x_2 + x_4,$

$f_3(x_1, x_2, x_3, x_4) = -2x_1 - 4x_3 + 3x_4.$

15) (a) Let V be a finite-dimensional vector space over the field F. For each vector α in V define $L_\alpha(f) = f(\alpha), f$ in V^*. Prove that the mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}.

. **(OR)**

(b) Let V and W be finite-dimensional vector spaces over the field F. Let $\mathfrak{B}$ be an ordered basis for V with dual basis $\mathfrak{B}^*$, and let $\mathfrak{B}'$ be an ordered basis for W with dual basis $\mathfrak{B}'^*$. Let T be a linear transformation from V into W; let A be the matrix of T relative to $\mathfrak{B}, \mathfrak{B}'$ and let B be the matrix of T^t relative to $\mathfrak{B}'^*, \mathfrak{B}^*$. Then prove that $B_{ij} = A_{ji}$.

SECTION – C (5 X 5 = 25 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS. (K4 (Or) K5)

16) (a) Prove that every $m \times n$ matrix A is row equivalent to a row reduced echelon matrix.

(OR)

(b) Discover whether $A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$ is invertible, and find A^{-1}.

17) (a) If W_1 and W_2 are finite dimensional subspace of a vector space V, then prove that $W_1 + W_2$ is infinite dimensional and

$$dim\, W_1 + dim\, W_2 = dim(W_1 + W_2) + dim\,(W_1 \cap W_2).$$

(OR)

(b) Let F be a subfield of the complex numbers. Then find the basis of P= $\begin{bmatrix} -1 & 4 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 8 \end{bmatrix}$ and also find the coordinates of each standard basis vectors.

18) a) Let V and W be vector space over the Field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional then prove that $rank\ (T) + nullity\ (T) = dim\, V.$

(OR)

(b) Prove that every n-dimensional vector space over the field F isomorphic to the space F^n.

(CONTD……3)

19) a) Let V be the space of all polynomial in variable x of degree at most 3 and let D be the differential operator and let $\mathfrak{B} = \{f_1, \dots \dots, f_n\}$ be the ordered basis for V defined by $f_i(x) = x^{i-1}$. Let A be the matrix of D w.r.t some basis for V, then find the matrix of A.

(OR)

(b) Let V be a finite-dimensional vector space over the field F, and let W be a subspace of V. Then prove that $\dim \mathrm{W} + \dim \mathrm{W}^0 = \dim \mathrm{V}$.

20) (a) If f is a non-zero linear functional on the vector space V, then prove that the null space of f is a hyperspace in V. Conversely, prove that every hyperspace in V is the null space of a (not unique) non-zero linear functional on V.

(OR)

(b) Let V and W be vector spaces over the field F, and let T be a linear transformation from V into W. The null space of T^t is the annihilator of the range of T. If V and W are finite-dimensional, then prove that

(i) rank (T^t) = rank (T)

(ii) the range T^t is the annihilator of the null space of T.

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