(FOR THE CANDIDATES ADMITTED

DURING THE ACADEMIC YEAR 2020 ONLY)

20PMS416

REG.NO. :

N.G.M.COLLEGE (AUTONOMOUS) : POLLACHI
END-OF-SEMESTER EXAMINATIONS : MAY-2024
COURSE NAME: M.Sc.-MATHEMATICS **MAXIMUM MARKS: 70**
SEMESTER: IV **TIME : 3 HOURS**

OPERATOR THEORY

SECTION – A (10 X 1 = 10 MARKS)

ANSWER THE FOLLOWING QUESTIONS.
MULTIPLE CHOICE QUESTIONS. (K1)

1. The total variation of $\phi, V(\phi) :=$ ________
   a) $\inf_{\pi} \sum_{j=1}^{n} |\phi(t_j) - \phi(t_{j-1})|$
   b) $\inf_{\pi} \sum_{j=1}^{n} |\phi(t_j) + \phi(t_{j-1})|$
   c) $\sup_{\pi} \sum_{j=1}^{n} |\phi(t_j) - \phi(t_{j-1})|$
   d) $\sup_{\pi} \sum_{j=1}^{n} |\phi(t_j) + \phi(t_{j-1})|$

2. If span S is dense in X', then $\tilde{S} =$ ___________.
   a) 0 b) 1 c) -1 d) ∞

3. If A is a linear operator is completely continuous operator, then $x_n \to x$ weekly $\Rightarrow$ ________.
   a) $Ax_n \to x$ b) $A \to Ax$ c) $Ax_n \to A$ d) $Ax_n \to Ax$

4. Eigen spectrum is also known as ___________
   a) Eigen vector b) Point spectrum c) Eigen value d) None of these

5. $\boldsymbol{r_w(A)} := \sup\{|\lambda| : \lambda \in w(A)\}$ is called _____ of A.
   a) Numerical Radius
   b) Numerical Value
   c) Numerical Range
   d) Numerical Equation

ANSWER THE FOLLOWING IN ONE OR TWO SENTENCES. (K2)

6. Define: K^n with $\|.\|_p$
7. Define: Absolutely Continuous
8. When will you say a normed linear space X is reflexive?
9. What is Ideal?.
10. Define: Eigen spectrum

SECTION – B (5 X 4 = 20 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS. (K3)

11. a) State the Fundamental Theorem of Lebesque Integration.

(OR)

b) Prove that a normed liner space is separable if its dual is separable.

12. a) Suppose X is reflexive. Then prove that X is separable iff its dual X' is separable.

(OR)

b) Let $E \subseteq X'$ be such that span E is dense in X', be a bounded sequence in X, and $x \in X$. Then prove that if $f(x_n) \to f(x)$ for every $f \in E$, then (x_n) converges weekly to x.

13. a) Prove that for every bounded sequence (x_n) in $L^1[a,b]$, the sequence (Kx_n) has a convergent subsequence in $(C[a,b], \|.\|_\infty)$.

(OR)

b) Let X and Y be normed linear spaces and $A: X \to Y$ be an injective compact operator. Then prove that if $A^{-1}: R(A) \to X$ is continuous iff $rank\ A < \infty$.

(CONTD…….2)

14.a) Let X be a normed linear space and $A: X \to X$ be a linear operator. Then prove that $\sigma_{eig}(A) \subseteq \sigma_{app}(A)$. If the space X is finite dimensional, then prove that $\sigma_{eig}(A) = \sigma_{app}(A)$.

(OR)

b) Suppose X is a Banach Space, $A \in \mathrm{B}(X) and\ \lambda \epsilon \mathrm{K}$. Then prove that $\lambda\ \epsilon \sigma(A)$ iff either $\lambda\ \epsilon \sigma_{app}(A) or\ R(A - \lambda I)$ is not dense in X.

15.a) Let X be a Hilbert space and Y be an inner product space. Then prove that every $A \in \mathrm{B}(X, Y)$ has the adjoint.

(OR)

b) Let X be a Hilbert space and $A \in \mathrm{B}(X)$ be a self adjoint operator. Then prove that $\| \quad \| = sup\{|\langle Ax, x\rangle| : x \in X, \|x\| = 1\}$.In particular, $A = 0$ iff $\langle Ax, x\rangle = 0, \forall\ x \in X$.

SECTION - C (4 X 10 = 40 MARKS)

ANSWER ANY FOUR OUT OF SIX QUESTIONS.

(16th QUESTION IS COMPULSORY AND ANSWER ANY THREE QUESTIONS FROM Qn.No : 17 to 21). (K4 (Or) K5)

16. a) Prove: The map $T: \left(l^p(n)\right)' \to l^q(n)$ defined by $T(f) = \left(f(e_1), \dots f(e_n)\right), f \in (l^p)'$ is surjective linear isometry, where $e_j \in \mathrm{K}_n$ is such that $e_j(i) = \delta_{ij}$ for $i, j = 1,2, \dots, n$.

(OR)

b) Let $1 \le p \le \infty$. For $y \in L^q[a, b]$, let $f_y: L^q[a, b] \to \mathrm{K}$ be defined by $f_y(x) := \int_a^b xy\, d\mu \ \ x \in L^q[a, b]$. Then prove that $f_y \in (L^q[a, b])'$ and the map $y \longmapsto f_y$ is a linear isometry from $L^q[a, b]$ to $(L^q[a, b])'$.

17. a) State and Prove Schur's Lemma.

(OR)

b) Suppose X is a reflexive space. Then prove that

i) If Y is a closed subspace of X, then for every $x \in X$, there exists $y \in Y$ such that $\|x - y\| = dist\ (x, Y)$.

ii) If Y is a proper subspace of X, then there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $dist\ (x_0, Y) = 1$.

iii) For every $f \in X'$, there exists $x \in X$ such that $\|x\| = 1, |f(x)| = \|f\|$.

18. a) Let X and Y be normed linear spaces and$: X \to Y$ be a linear operator. Then prove that the following are equivalent:

i) A is a compact operator.

ii) $cl\{Ax: \|x\| < 1\}$ is compact in Y.

iii) For every bounded subset E of X, $cl\ A(E)$ is compact in Y.

iv) For every bounded sequence (x_n) in X, the sequence (Ax_n) has a convergent subsequence in Y.

(OR)

b) Let X and Y be normed linear spaces and $X_0 \subseteq N(A)$ be a closed subspace of X. If $A: X \to Y$ is a compact operator, then prove that $A: X/X_0 \to Y$ is also a compact operator.

19. a) Let X be a Banach Space, and $A \in \mathrm{B}(X)$. If $\|A\| < 1$, then prove that $I - A$ is invertible and $\|(I - A)^{-1}\| \le \frac{1}{1 - \|A\|}$.

(OR)

b) Let X be a Banach Space, and $A \in \mathrm{B}(X)$. Then prove that every boundary point of $\sigma(A)$ is an approximate eigenvalue of A.

20.a) State and prove Closed Range Theorem.

(OR)

b) Let X be a Hilbert space and $A \in \mathrm{B}(X)$. Then prove that

i) A is normal iff $\|Ax\| = \|A^*x\|$, for every $x \in X$.

ii) A is unitary iff A is surjective and $\|Ax\| = \|x\|$, for every $x \in X$.

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