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(FOR THE CANDIDATES ADMITTED

DURING THE ACADEMIC YEAR 2021 ONLY)

21PMS205

REG.NO. :

N.G.M.COLLEGE (AUTONOMOUS) : POLLACHI

END-OF-SEMESTER EXAMINATIONS : JULY -2022

M.Sc.-MATHEMATICS **MAXIMUM MARKS: 70**

SEMESTER II **TIME : 3 HOURS**

LINEAR ALGEBRA

SECTION - A (10 X 1 = 10 MARKS)

ANSWER THE FOLLOWING QUESTIONS.

MULTIPLE CHOICE QUESTIONS. **(K1)**

1. The characteristic value of the matrix A = $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ______________.
   a) 2 b) no values c) 3 d) 1
2. The unique expression for α as a sum of vectors in R and N is ____________.
   a) α =Eα + (α-Eα) b) α =Eα - (α-Eα) c) α =Eα + (α +Eα) d) α = E (Eα)+ α
3. The identity operator has no cyclic vector, when_________________.
   a) dim V >1 b) dim V = 0 c) dim V < 1 d) dim V = 1
4. Every entry of A not on or immediately below the main diagonal is
   a) 1 b) 2 c) 3 d) 0
5. rank f = _____________.
   a) dim V^+ + dim V^- b)dim V^+ - dim V^- c) dim V^+ + dim V a) dim V + dim V^-

ANSWER THE FOLLOWING IN ONE (OR) TWO SENTENCES. **(K2)**

6. When do you say that the linear operator T is diagonalizable?
7. Define nilpotent linear operator.
8. Define the T -cyclic subspace generated by any vector.
9. What are the conditions for the matrix N to be in normal form?
10. When is a bilinear form f on vector spaces called non – degenerate?

SECTION – B (5 X 4 = 20 MARKS)

ANSWER EITHER (a) OR (b) IN EACH OF THE FOLLOWING QUESTIONS. **(K3)**

11. a) Let T be a linear operator on a finite dimensional space V. Let $c_1,\ldots c_k$ be the distinct characteristic values of T and let W_i be the null space of $(T - c_iI)$. Prove that the following are equivalent
   (i) T is diagonalizable.
   (ii) The characteristic polynomial for T is $f = (x - c_1)^{d_1} \ldots (x - c_k)^{d_k}$ and dim $W_i = d_i$, i = 1,…k.
   (iii) dim W_1+ . . .+dim W_k = dim V.

(OR)

(CONTD….2)

b) Let V be a finite dimensional vector space over the field F. Let T be a linear operator on V such that the minimal polynomial for T is a product of linear factors $p = (x - c_1)^{r1} \ldots (x - c_k)^{rk}$, c_i in F. Let W be a proper ($W \neq V$) subspace of V which is invariant under T. Prove that there exist a vector α in V such that
(i) α is not in W (ii) $(T - cI)\alpha$ is in w, for some characteristic value c of the operator T.

12.a) If $V = W_1 \oplus \ldots \oplus W_k$, prove that there exist k linear operators $E_1,\ldots,E_k$ on V such that
(i) each E_i is a projection ($E_i^2 = E^i$);
(ii)$E_iE_j = 0$, if $i \neq j$;
(iii)$I = E_1+\ldots+E_k$;
(iv)the range of E_i is W_i
Is the converse true? Justify

(OR)

b) Let T be a linear operator on the finite dimensional vector space V over the field F. Suppose that the minimal polynomial for T decomposes over F into a product of linear polynomials. Prove that there is a diagonalizable operator D on V and a nilpotent operator N on V such that (i) T = D + N (ii)DN = ND, . where the diagonalizable operator D and the Nilpotent operator N are uniquely determined by (i) and (ii) and each of them is a polynomial in T.

13.a) Let α is an non-zero vector in V and let p_α be the T – annihilator of α. Prove the following:
(i) The degree of p_α is equal to the dimension of the cyclic subspace Z(α;T).
(ii)If the degree of p_α is k, then the vectors$\{\alpha, T\alpha, T^2\alpha, \ldots T^{k-1}\alpha\}$ form a basis for Z(α;T).
(iii)If U is the linear operator on Z(α;T)induced by T, then the minimal polynomial for U is p_α

(OR)

b) If F be a field and B is an n x n matrix over F, prove that B is similar over the field F to one and only one matrix, which is in rational form.

14.a) Let A be a n x n matrix with entries in the field F , and let $p_1,\ldots p_r$ be the invariant factors for a. The prove that the matrix xI –A is equivalent to the n x n diagonal matrix with diagonal entries $p_1,\ldots p_r,1,1,\ldots 1$

(OR)

b) If M and N are equivalent m x n matrices with entries in F[x], then prove that $\delta_k(M) = \delta_k(N)$, $1 \leq k \leq \min(m,n)$.

15.a) Let f be a bilinear form on the finite dimensional vector space V. and L_f and R_f be the linear transformations from V into V^* defined by $(L_{f\alpha})(\beta) = f(\alpha,\beta) = (R_f\beta)(\alpha)$. Prove that $(L_f) = \text{rank}(R_f)$

(OR)

b) Let V be a finite – dimensional vector space over a field of characteristic zero, and f be a symmetric bilinear form on V. Prove that there is an ordered basis for V in which f is represented by a diagonal matrix.

SECTION - C (4 X 10 = 40 MARKS)

ANSWER ANY FOUR OUT OF SIX QUESTIONS
(16^{th} QUESTION IS COMPULSORY AND ANSWER ANY THREE QUESTIONS (FROM Qn. No : 17 to 21) (K4 (Or) K5)

16. State and Prove the Primary Decomposition Theorem.

17. State and Prove Cayley – Hamilton Theorem.

18. Let T be a linear operator on the finite dimensional vector space V. If T is diagonalizable and if $c_1,\ldots c_k$ are the distinct characteristic values of T, prove that there exist linear operators $E_1,\ldots,E_k$ on V such that (i) $T = c_1 E_1,\ldots, c_k E_k$;(ii) $I = E_1+\ldots+E_k$; (iii) $E_iE_j = 0$, if $i \neq j$; (iv) $E_i^2 = E_i$ (v) the range of E_i is the characteristic space for T associated with c_i. Prove the converse also.

19. State and Prove Cyclic Decomposition Theorem.

20. Let P be an m $\times$ m matrix with entries in the polynomial algebra F[X]. Prove that the following are equivalent
   i) P is invertible.
   ii) The determinant of P is a non-zero scalar polynomial.
   iii) P is row equivalent to the m $\times$ m identity matrix.
   iv) P is a product of elementary matrices.

21. Let V be an n-dimensional vector space over the field of real numbers and let f be a symmetric bilinear form on V which has rank r. Prove that there is an ordered basis $\{\beta 1, \beta 2 ,..., \beta n\}$ for V in which the matrix of f is diagonal and such that
   $f(\beta j, \beta j) = \pm 1, \quad j = 1, \ldots, r.$
   Prove further then, the number of basis vectors βj, for which $(\beta j, \beta j) = 1$ is independent of the choice of basis.

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